

# RELATIONS IN THE 2-CLASS GROUP OF QUADRATIC NUMBER FIELDS

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ABSTRACT. We construct a family of ideals representing ideal classes of order 2 in quadratic number fields and show that relations between their ideal classes are governed by certain cyclic quartic extensions of the rationals.

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## INTRODUCTION

Let  $m = p_1 \cdots p_t$  be a product of pairwise distinct primes  $p_j \equiv 1 \pmod{4}$ . Then  $m$  can be written as a sum of two squares, say  $m = a_j^2 + 4b_j^2$ , in  $2^{t-1}$  essentially different ways (i.e., neglecting the signs of  $a_j$  and  $b_j$ ).

We define ideals  $\mathfrak{a}_j = (2b_j + \sqrt{m}, a_j)$  in the ring of integers of the quadratic number field  $K = \mathbb{Q}(\sqrt{m})$ . Since

$$\begin{aligned} \mathfrak{a}_j^2 &= ((2b_j + \sqrt{m})^2, a_j(2b_j + \sqrt{m}), a_j^2) \\ &= ((2b_j + \sqrt{m})^2, a_j(2b_j + \sqrt{m}), m - 4b_j^2) \\ &= (2b_j + \sqrt{m})(2b_j + \sqrt{m}, a_j, 2b_j - \sqrt{m}) \\ &= (2b_j + \sqrt{m})(4b_j, a_j, 2b_j - \sqrt{m}) = (2b_j + \sqrt{m}) \end{aligned}$$

is principal, the ideals  $\mathfrak{a}_j$  have order dividing 2.

The "canonical" ideals generating classes of order dividing 2 are the products of ramified prime ideals. Letting  $\mathfrak{p}_j$  denote the prime ideal above the prime  $p_j$ , we have  $\mathfrak{p}_j^2 = (p_j)$ . Thus each of the  $2^t$  ideals

$$\mathfrak{b}_e = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}, \quad e = (e_1, \dots, e_t) \in \mathbb{F}_2^t,$$

generates a class of order dividing 2. Among these ideal classes there are two trivial relations coming from the fact that  $(1) = \prod \mathfrak{p}_j^0$  and  $(\sqrt{m}) = \prod \mathfrak{p}_j^1$  are principal ideals (in the usual sense; the ideal class of  $(\sqrt{m})$  is principal in the strict sense if and only if the fundamental unit  $\varepsilon$  of  $K$  has norm  $-1$ ).

The following result is well known:

**Proposition 1.** *There is a nontrivial relation among the ideal classes of the ideals  $\mathfrak{b}_e$  if and only if the norm of the fundamental unit  $\varepsilon$  of  $K$  is  $+1$ .*

*Proof.* If  $\prod \mathfrak{p}_j^{e_j} = (\alpha)$  is principal, then  $(\alpha)^2 = \prod p_j^{e_j}$ , hence  $\eta = \alpha^2 / \prod p_j^{e_j}$  is a unit with norm  $+1$ . If  $\eta$  is a square, then so is  $\prod p_j^{e_j}$ , which is only possible for  $e = (0, \dots, 0)$  and  $e = (1, \dots, 1)$ . Thus if there is a nontrivial relation among the classes of the ramified ideals, then there is a nonsquare unit with positive norm; this implies that  $N\varepsilon = +1$ .

Conversely, if  $N\varepsilon = +1$ , then  $\varepsilon = \alpha^{1-\sigma}$  for some  $\alpha \in \mathcal{O}_K$  by Hilbert 90, where  $\sigma$  is the nontrivial automorphism of  $K/\mathbb{Q}$ . Since  $(\alpha)$  is fixed by the Galois group of

$K/\mathbb{Q}$ , the ideal  $(\alpha)$  is a product of a rational integer and a ramified ideal. Cancelling the rational factors we see that we may assume that  $(\alpha)$  is a product of ramified prime ideals. The equations  $\alpha = 1$  and  $\alpha = \sqrt{m}$  would imply  $\varepsilon = \pm 1$ ; thus the relation  $(\alpha) = \prod \mathfrak{p}_j^{e_j}$  is necessarily nontrivial.  $\square$

The goal of this article is to clarify the relations between the ideals  $\mathfrak{a}_j$  and  $\mathfrak{b}_e$ . Our main result is the following

**Theorem 1.** *Let  $K = \mathbb{Q}(\sqrt{m})$  be a quadratic number field, where  $m = p_1 \cdots p_t$  is a product of primes  $p_j \equiv 1 \pmod{4}$ . Let  $\varepsilon$  denote the fundamental unit of  $K$ .*

- a) *If  $N\varepsilon = -1$ , then the ideal classes  $[\mathfrak{a}_j]$  are pairwise distinct and represent the  $2^{t-1}$  classes of order dividing 2 in  $\text{Cl}(K)$ . Each ideal  $\mathfrak{a}_j$  is equivalent to a unique ramified ideal  $\mathfrak{b}_e$ . In particular, exactly one of the  $\mathfrak{a}_j$  is principal; if  $\mathfrak{a}_j = (\alpha)$ , then*

$$\eta = \frac{2b_j + \sqrt{m}}{\alpha^2}$$

*is a unit with norm  $-1$  (equal to  $\varepsilon$  if  $\alpha$  is chosen suitably).*

- b) *If  $N\varepsilon = +1$ , then there is a subgroup  $C$  with index 2 in the group  $\text{Cl}(K)[2]$  of ideal classes of order dividing 2 such that each class in  $C$  is represented by two ramified ideals  $\mathfrak{b}_e$  (thus  $C$  is the group of strongly ambiguous ideal classes in  $K$ ). Each class in  $\text{Cl}(K)[2] \setminus C$  is represented by two ideals  $\mathfrak{a}_j$ .*

The proof of Thm. 1 uses certain quadratic extensions of  $\mathbb{Q}(\sqrt{m})$ , namely cyclic quartic subextensions of the field  $\mathbb{Q}(\zeta_m)$  of  $m$ -th roots of unity. It is perhaps surprising that relations in the class group of  $K$  are governed by *ramified* extensions of  $K$ ; note, however, that if  $K_1$  and  $K_2$  are two cyclic quartic extensions as above, then the compositum  $K_1 K_2$  contains a third quadratic extension  $K_3/K$ , and that this extension is unramified: in fact, it is part of the genus class field of  $K$ .

Here is an example. Let  $m = 5 \cdot 13 \cdot 29 = 1885$ ; then  $m = 6^2 + 43^2 = 11^2 + 42^2 = 21^2 + 38^2 = 27^2 + 34^2$ , and we consider the ideals  $\mathfrak{a}_1 = (6 + \sqrt{m}, 43)$ ,  $\mathfrak{a}_2 = (42 + \sqrt{m}, 11)$ ,  $\mathfrak{a}_3 = (38 + \sqrt{m}, 21)$ ,  $\mathfrak{a}_4 = (34 + \sqrt{m}, 27)$ . Since  $N\varepsilon = +1$ , none of these ideals is principal. In fact we have  $\mathfrak{a}_2 \sim \mathfrak{a}_3$  and  $\mathfrak{a}_1 \sim \mathfrak{a}_4$ . If  $\mathfrak{p}$  denotes the ideal with norm 5, then the ideal classes of order 2 are represented by  $\mathfrak{p}$ ,  $\mathfrak{a}_1$  and  $\mathfrak{a}_2 \sim \mathfrak{a}_1 \mathfrak{p}$ .

The ramified prime ideal above 29 is principal (in fact,  $N(87 + 2\sqrt{1885}) = 29$ ), those above 5 and 13 are not. In particular,  $\sqrt{\varepsilon} = 2\sqrt{65} + 3\sqrt{29}$ .

## 1. CYCLIC QUARTIC EXTENSIONS

Let  $m = a^2 + 4b^2$  be a squarefree odd integer; then it is the discriminant of the quadratic number field  $k = \mathbb{Q}(\sqrt{m})$ . In the following, we will always assume that  $m = p_1 \cdots p_t$ , where the  $p_j \equiv 1 \pmod{4}$  are prime numbers.

Some basic facts concerning the description of abelian extensions of the rationals via characters can be found in [2]. The field of  $m$ -th roots of unity has Galois group isomorphic to  $(\mathbb{Z}/m\mathbb{Z})^\times \simeq \prod (\mathbb{Z}/p_j\mathbb{Z})^\times$ , hence has  $G = (\mathbb{Z}/4\mathbb{Z})^t$  as a quotient. This group  $G$  is the Galois group of the compositum  $F$  of the cyclic quartic extensions inside the fields  $\mathbb{Q}(\zeta_{p_j})$ . The character group  $X = X(G)$  is generated by quartic Dirichlet characters  $\chi_j = \chi_{p_j}$ , and the cyclic quartic subfields of  $L$  (and of  $\mathbb{Q}(\zeta_m)$ ) correspond to cyclic subgroups of  $X$  with order 4.

Let  $\chi$  be a character generating such a cyclic group of order 4, and let  $L/\mathbb{Q}$  be the corresponding cyclic quartic extension. Then  $\chi = \prod \chi_j^{e_j}$  with  $0 \leq e_j < 4$ . The

order of  $\chi$  is 4 if and only if at least one exponent  $e_j$  is odd. By the conductor-discriminant formula, the field  $L$  has discriminant  $m^3$  if and only if all the  $e_j$  are odd. Thus these characters correspond to vectors  $e = (e_1, \dots, e_t)$  with  $e_j \in \{1, 3\}$ . The cyclic quartic extensions  $L/\mathbb{Q}$  inside  $F$  with discriminant  $m^3$  correspond to subgroups generated by such characters, and since each subgroup contains two such characters ( $\chi$  and  $\chi^3$ ), we have

**Proposition 2.** *Let  $m = p_1 \cdots p_t$  be a product of distinct primes  $p_j \equiv 1 \pmod{4}$ . Then there are  $2^{t-1}$  cyclic quartic extensions  $L/\mathbb{Q}$  with conductor  $m$  and discriminant  $m^3$ .*

These extensions can be constructed explicitly:

**Proposition 3.** *Let  $m = p_1 \cdots p_t$  be a product of pairwise distinct primes  $p \equiv 1 \pmod{4}$ . Then there exist  $2^{t-1}$  ways of writing  $m = a_j^2 + 4b_j^2$  as a sum of two squares (up to sign). For each  $j$ , the extensions*

$$L = \mathbb{Q}(\sqrt{m + 2b_j\sqrt{m}})$$

*are the  $2^{t-1}$  different cyclic quartic extensions of  $\mathbb{Q}$  with discriminant  $m^3$ .*

We will prove this result below; now we will use it for giving the

*Proof of Theorem 1.* Assume first that  $N\varepsilon = -1$ . We have to show that the classes of the ideals  $\mathfrak{a}_j = (2b_j + \sqrt{m}, a_j)$  are pairwise distinct.

Assume to the contrary that  $\mathfrak{a}_j \sim \mathfrak{a}_k$ ; then there exists some  $\xi \in K$  with  $\mathfrak{a}_j = \xi \mathfrak{a}_k$ . Squaring gives the equation  $(2b_j + \sqrt{m}) = \xi^2(2b_k + \sqrt{m})$  of ideals, hence there exists a unit  $\eta \in \mathcal{O}_K^\times$  such that  $2b_j + \sqrt{m} = \eta \xi^2(2b_k + \sqrt{m})$ . This means that the square roots of  $(2b_j + \sqrt{m})\sqrt{m}$  and  $\eta(2b_k + \sqrt{m})\sqrt{m}$  must generate the same extension. Since  $\mathbb{Q}(\sqrt{(2b_j + \sqrt{m})\sqrt{m}})$  is a cyclic quartic extension inside  $\mathbb{Q}(\zeta_m)$ , so is the extension on the right hand side. But this implies that  $N\eta = +1$ , and the fact that  $N\varepsilon = -1$  implies that  $\eta = \pm \varepsilon^{2n}$ . Subsuming the unit into  $\xi$  shows that we may assume that  $\eta = \pm 1$ . If  $\eta = -1$ , the extension on the right hand side will ramify at 2, and this finally shows that  $\eta$  is a square. Thus we have  $\mathbb{Q}(\sqrt{(2b_j + \sqrt{m})\sqrt{m}}) = \mathbb{Q}(\sqrt{(2b_k + \sqrt{m})\sqrt{m}})$  and this can only hold if  $j = k$ .

We have shown that exactly one ideal  $\mathfrak{a}_j$  is principal, say  $\mathfrak{a}_j = (\alpha)$ . Since  $(\alpha)^2 = \mathfrak{a}_j^2 = (2b_j + \sqrt{m})$  there exists a unit  $\eta$  such that  $\eta\alpha^2 = 2b_j + \sqrt{m}$ . Taking the norm shows that  $N\eta = -1$  as claimed.

If  $N\varepsilon = +1$ , on the other hand, we first show that none of the ideals  $\mathfrak{a}_j$  is equivalent to a ramified ideal  $\mathfrak{b}_e$ . In fact, if  $\mathfrak{a}_j = \xi \mathfrak{b}_e$  for some  $\xi \in K^\times$ , then squaring yields  $2b_j + \sqrt{m} = \eta \xi^2 m_1$  for  $m_1 = \prod p_j^{e_j}$  and some unit  $\eta$ . Taking norms shows that  $N\eta = -1$ , which contradicts our assumptions.

Next we show that among the classes of  $\mathfrak{a}_j$ , each ideal class occurs twice. In fact, if  $\mathfrak{a}_j \sim \mathfrak{a}_k$ , say  $\mathfrak{a}_j = \xi \mathfrak{a}_k$ , then  $2b_j + \sqrt{m} = \eta \xi^2(2b_k + \sqrt{m})$ . Up to squares,  $\eta$  is equal to one of the units  $\pm 1, \pm \varepsilon$ . As above,  $\eta = -1$  and  $\eta = -\varepsilon$  are impossible, since the places at infinity ramify in the extension  $K(\sqrt{\eta(2b_k + \sqrt{m})\sqrt{m}})/K$  but not in  $K(\sqrt{(2b_j + \sqrt{m})\sqrt{m}})/K$ . Thus either  $\eta = 1$  and  $j = k$ , or  $\eta = \varepsilon$ .

Thus each  $\mathfrak{a}_j$  is equivalent to at most one other  $\mathfrak{a}_k$ . Since there are  $2^{t-1}$  ideals  $\mathfrak{a}_j$ , which are distributed among the  $2^{t-1}$  ideal classes of order 2 in  $\text{Cl}(K)[2] \setminus C$ , it follows that each  $\mathfrak{a}_j$  is equivalent to exactly one other  $\mathfrak{a}_k$  as claimed.  $\square$

## 2. GENERATORS OF CYCLIC QUARTIC EXTENSIONS

In this section we will give a proof of Prop. 3.

**Kummer generators over  $\mathbb{Q}(i)$ .** Cyclic quartic extensions  $L/\mathbb{Q}$  become Kummer extensions over  $\mathbb{Q}' = \mathbb{Q}(i)$ : with  $L' = L(i)$  we have  $L' = \mathbb{Q}'(\sqrt[4]{\alpha})$  for some  $\alpha \in \mathbb{Q}(i)$ . Such an extension  $L'$  will be normal over  $\mathbb{Q}$  if and only if  $\alpha^{1-\sigma} = \alpha_\sigma^2$  is a square in  $\mathbb{Q}'$ , where  $\sigma$  is the nontrivial automorphism of  $\mathbb{Q}'/\mathbb{Q}$ , and will be abelian over  $\mathbb{Q}$  if and only if  $\sigma$  acts on  $\alpha_\sigma$  as on a fourth root of unity, i.e., if and only if  $\alpha_\sigma^\sigma = \alpha_\sigma^{-1}$ .

Since  $\alpha_\sigma^{\sigma+1} = 1$ , Hilbert 90 implies that  $\alpha_\sigma = \mu^{\sigma-1} = \bar{\mu}/\mu$  for some  $\mu \in \mathbb{Z}[i]$ . Thus  $\alpha^{1-\sigma} = (\bar{\mu}/\mu)^2$ . This equation is solved by  $\alpha = \mu\bar{\mu}^3 = m\bar{\mu}^2$ , where  $m = \mu\bar{\mu}$  is a positive integer. Galois theory actually shows that this is the only solution up to multiplying  $\alpha$  by some nonzero rational number.

We claim that  $\alpha$  can be chosen in such a way that 2 is unramified in  $L'/\mathbb{Q}'$ . Since  $k' = \mathbb{Q}'(\sqrt{\alpha}) = \mathbb{Q}'(\sqrt{m})$ , the prime above 2 does not ramify in the quadratic subextension  $k'/\mathbb{Q}'$ . In order that 2 be unramified in  $L'/k'$  we have to make sure that  $\sqrt{\alpha} = \bar{\mu}\sqrt{m} \equiv \mu\sqrt{m} \pmod{4}$  is congruent to a square modulo 4 in  $k'$  (see [1] for the decomposition law in quadratic and, more generally, Kummer extensions of prime degree). If we write  $\mu = a + 2bi$ , then  $\mu \equiv \pm 1 + 2i \pmod{4}$  if  $m \equiv 5 \pmod{8}$ , hence

$$\left(\frac{\pm 1 + 2i + i\sqrt{m}}{1+i}\right)^2 \equiv 2 + (\pm 1 + 2i)\sqrt{m} \equiv (\mp 1 + 2i)\sqrt{m} \pmod{4}$$

since  $2 \equiv 2\sqrt{m} \pmod{4}$ . Similarly, we have  $\mu \equiv \pm 1 \pmod{4}$  if  $m \equiv 1 \pmod{8}$ , hence

$$\left(\frac{\pm 1 + i\sqrt{m}}{1+i}\right)^2 \equiv \sqrt{m} \equiv \bar{\mu}\sqrt{m} \pmod{4}.$$

Thus 2 is unramified in  $L'/k'$  if we choose  $\mu \equiv 1 \pmod{2}$ .

We have proved

**Lemma 1.** *If  $L/\mathbb{Q}$  is a cyclic quartic extension with conductor  $m$ , then there exist integers  $a, b$  such that  $L' = \mathbb{Q}'(\sqrt[4]{\alpha})$  for  $\alpha = \mu\bar{\mu}^3$ , where  $\mu = a + 2bi \in \mathbb{Z}[i]$  and  $m = a^2 + 4b^2$ . Replacing  $a$  by  $-a$  or  $b$  by  $-b$  does not change the extension.*

**Kummer Generators over  $\mathbb{Q}$ .** Let  $L' = \mathbb{Q}'(\sqrt[4]{\alpha})$  be a Kummer extension of degree 4 over  $\mathbb{Q}' = \mathbb{Q}(i)$ , and assume that  $\alpha = \mu\bar{\mu}^3$  for some  $\mu = a + 2bi \in \mathbb{Z}[i]$ . Set  $\beta = \sqrt[4]{\alpha}$ ; we claim that  $\beta + \beta'$  is an element of  $\mathbb{Q}(\sqrt{m})$ , where  $m = \mu\bar{\mu} = a^2 + 4b^2$ . In fact,  $(\beta + \beta')^2 = \sqrt{m}(\mu + \bar{\mu}) + 2m = 2m + 2a\sqrt{m}$ . This implies that  $L'$  contains a quartic subextension  $L = \mathbb{Q}(\sqrt{2m + 2a\sqrt{m}})$ .

The following lemma shows that  $L$  is also generated by  $\sqrt{m + 2b\sqrt{m}}$ :

**Lemma 2.** *Assume that  $Ax^2 - By^2 - Cz^2 = 0$ . Then*

$$(1) \quad 2(x\sqrt{A} + y\sqrt{B})(x\sqrt{A} + z\sqrt{C}) = (x\sqrt{A} + y\sqrt{B} + z\sqrt{C})^2.$$

*Proof.* We have

$$\begin{aligned} (x\sqrt{A} + y\sqrt{B} + z\sqrt{C})^2 &= Ax^2 + By^2 + Cz^2 + 2xy\sqrt{AB} + 2xz\sqrt{AC} + 2yz\sqrt{BC} \\ &= 2(Ax^2 + xy\sqrt{AB} + xz\sqrt{AC} + yz\sqrt{BC}) \\ &= 2(x\sqrt{A} + y\sqrt{B})(x\sqrt{A} + z\sqrt{C}) \end{aligned}$$

as claimed.  $\square$

Since  $m - a^2 - 4b^2 = 0$ , the lemma shows that  $2(\sqrt{m} + a)(\sqrt{m} + 2b)$  is a square in  $K = \mathbb{Q}(\sqrt{m})$ , hence  $2b\sqrt{m} + m$  and  $2m + 2a\sqrt{m}$  generate the same quadratic extension of  $K$ . This finishes the proof of Prop. 3.

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